Solvable aggregation-annihilation processes with greater than two components

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We study the aggregation-annihilation processes involving three or more kinds of distinct monomers species based on mean-field theory. We propose several solvable models in which irreversible bonding occurs only between similar species and annihilation occurs only between dissimilar species. Under constant aggregation and annihilation reaction rates, the exact solutions of these models are obtained. These analytical solutions show that the kinetic evolution behaviors of each species are quite different. They are scaling or nonscaling depending conclusively on all reaction rates and initial mass distribution of each of the reactants.
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I. INTRODUCTION

Aggregation processes are of widespread interest in many branches of physics, chemistry, and biology. Typically, aggregation processes can be described by the reaction scheme [1–3,23]

$$A_i + A_j \xrightarrow{K(i,j)} A_{i+j}.$$
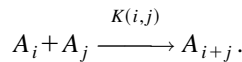
Here A_i is a cluster consisting of i monomers, the A_i cluster and A_j cluster aggregate together with reaction rate $K(i,j)$, resulting a larger cluster A_{i+j}. In recent years, the kinetic investigation of aggregation processes have made considerable developments [4–12]. However, most of the research works focus on the aggregation, annihilation, and fragmentation for one species [3,13–16], and an exact solution for that species, in one dimension, has been obtained [17–22]. Few research works consider the aggregation-annihilation for two species [22–24]. In this paper, we study the aggregation-annihilation processes in a diffusion-limited reaction with $l > 2$ species. These processes may occur in many systems of physics, chemistry, and biology, such as the processes of dress ore consisting of many kinds of elements, the organic chemical reactions forming compound material, as well as the processes of leavening with lots of fungi. Because the solution of this problem is much more complicated than that for one and two species, we propose several solvable simplified models. In our model, according to [23], the irreversible aggregation reaction occurs only between similar species $A_i + A_j \rightarrow A_{i+j}$, and the annihilation reaction occurs only between dissimilar species $A_{i+j} + B_i \rightarrow A_j$, $A_i + B_{i+j} \rightarrow B_j$. In general, an annihilation reaction may occur between any two distinct species. We especially consider the following interesting special case: if various distinct species are denoted by A^l, the annihilation reaction occurs only between species A^l and $A^{l\pm1}$ ($l = 1,2,3,...$). According to our work, it is found that under constant reaction rates of aggregation and annihilation the kinetic evolution behaviors of reactants crucially depend on all reaction rates and initial mass distributions of each reactants.

The outline of this paper is as follows. In Sec. II, we discuss a special three-species aggregation-annihilation model in which the annihilation reaction occurs between species A^1 and A^2, as well as between species A^2 and A^3, and there is no annihilation reaction between species A^1 and A^3. We also give some special solutions. In Sec. III, we describe a many species aggregation-annihilation model.

II. THREE-SPECIES AGGREGATION-ANNIHILATION MODEL

In this model, our investigation is based on the mean-field theory and thus the spatial fluctuations of reactants are ignored. We have supposed three kinds of species A^1, A^2, and A^3, each species coalesces with a constant aggregation rate and annihilation processes occur only between A^1 and A^2, as well as between A^2 and A^3. For simplicity, we set the rate of aggregation processes the same for each species and equals one, and the rates of annihilation processes are all equal to J. If $a_{1k}(t)$, $a_{2k}(t)$, and $a_{3k}(t)$ denote, respectively, the concentrations of A^1-, A^2-, and A^3-clusters consisting of k monomers, the corresponding rate equations for this aggregation-annihilation system can be written as [23]

$$\frac{da_{1k}}{dt} = \sum_{i+j=k} a_{1i}, a_{1j} - 2a_{1k} \sum_{j=1}^{\infty} a_{1j}$$
$$+ J\left(\sum_{p=q+k} a_{1p} a_{2q} - a_{1k} \sum_{j=1}^{\infty} a_{2j} \right), \quad (1)$$

$$\frac{da_{2k}}{dt} = \sum_{i+j=k} a_{2i} a_{2j} - 2a_{2k} \sum_{j=1}^{\infty} a_{2j} + J\left(\sum_{p=q+k} a_{2p} a_{1q} \right.$$
$$\left. - a_{2k} \sum_{j=1}^{\infty} a_{1j} \right) + J\left(\sum_{p+q=k} a_{2p} a_{3q} - a_{2k} \sum_{j=1}^{\infty} a_{3j} \right),$$
$$(2)$$

$$\frac{da_{3k}}{dt} = \sum_{i+j=k} a_{3i}a_{3j} - 2a_{3k}\sum_{j=1}^{\infty} a_{3j}$$

$$+ J\left(\sum_{p=q+k} a_{3p}a_{3q} - a_{3k}\sum_{j=1}^{\infty} a_{2j} \right). \tag{3}$$

For the monodisperse initial conditions

$$a_{lk}(0) = A_{l0}\delta_{k1}, \quad l = 1,2,3, \tag{4}$$

these initial conditions mean that for each species there only exists one-monomer cluster whose concentration equals A_{l0} at $t=0$; the rate equations can be solved by means of an ansatz [1]

$$a_{lk} = A_l(a_l)^{k-1}, \quad l = 1,2,3 \tag{5}$$

where $A_l(t)$ and $a_l(t)$ are dependent on time. Utilizing ansatz (5), we rewrite Eqs. (1)–(3) as follows:

$$\frac{da_1}{dt} = A_1, \tag{6}$$

$$\frac{dA_1}{dt} = -\frac{2A_1^2}{1-a_1} + JA_1A_2\left(\frac{a_1}{1-a_1a_2} - \frac{1}{1-a_2} \right), \tag{7}$$

$$\frac{da_2}{dt} = A_2, \tag{8}$$

$$\frac{dA_2}{dt} = -\frac{2A_2^2}{1-a_2} + JA_2A_1\left(\frac{a_2}{1-a_2a_1} - \frac{1}{1-a_1} \right)$$

$$+ JA_2A_3\left(\frac{a_2}{1-a_2a_3} - \frac{1}{1-a_3} \right), \tag{9}$$

$$\frac{da_3}{dt} = A_3, \tag{10}$$

$$\frac{dA_3}{dt} = -\frac{2A_3^2}{1-a_3} + JA_3A_2\left(\frac{a_3}{1-a_3a_2} - \frac{1}{1-a_2} \right). \tag{11}$$

The initial conditions corresponding to Eq. (4) are rewritten as

$$a_l = 0, \quad A_l = \frac{da_l}{dt} = A_{l0}, \quad l = 1,2,3 \quad \text{at } t = 0. \tag{12}$$

Introducing new variables

$$\alpha_l = (1-a_l)^{-1}, \quad l = 1,2,3, \tag{13}$$

Eqs. (6)–(11) become

$$\frac{d^2\alpha_1}{dt^2} = -\frac{J}{\alpha_1+\alpha_2-1}\frac{d\alpha_1}{dt}\frac{d\alpha_2}{dt}, \tag{14}$$

$$\frac{d^2\alpha_2}{dt^2} = -\frac{J}{\alpha_2+\alpha_1-1}\frac{d\alpha_2}{dt}\frac{d\alpha_1}{dt} - \frac{J}{\alpha_2+\alpha_3-1}\frac{d\alpha_2}{dt}\frac{d\alpha_3}{dt}, \tag{15}$$

$$\frac{d^2\alpha_3}{dt^2} = -\frac{J}{\alpha_3+\alpha_2-1}\frac{d\alpha_3}{dt}\frac{d\alpha_2}{dt}, \tag{16}$$

and corresponding initial conditions

$$\alpha_l = 1, \quad \frac{d\alpha_l}{dt} = A_{l0}, \quad l = 1,2,3 \quad \text{at } t = 0. \tag{17}$$

Equations (14)–(16) are what we want to solve.

Obviously, there are two integrals from Eqs. (14)–(16); they are, respectively,

$$\alpha_1 + \alpha_3 - \alpha_2 = (A_{10}+A_{30}-A_{20})t + 1, \tag{18}$$

$$\frac{d\alpha_1}{dt}\frac{d\alpha_2}{dt}\frac{d\alpha_3}{dt} = A_{10}A_{20}A_{30}(\alpha_1+\alpha_2-1)^{-J}(\alpha_2+\alpha_3-1)^{-J}. \tag{19}$$

In order to find an explicit solution for α_1, α_2, and α_3, we consider some special symmetrical cases. These cases can be solved analytically.

(a) $\alpha_1 = \alpha_3 = \frac{1}{2}\alpha_2$. In this case, we assume $a_1 = a_3 = 2a_2 - 1$, $A_1 = A_3 = 2A_2$ and set the initial data

$$\alpha_1 = \alpha_3 = 1, \quad \alpha_2 = 2, \quad A_{10} = A_{30} = 2A_{20},$$

$$\frac{d\alpha_1}{dt} = A_{10}, \quad \frac{d\alpha_3}{dt} = A_{30}, \quad \frac{d\alpha_2}{dt} = 4A_{20} \quad \text{at } t = 0. \tag{20}$$

Substituting $\alpha_1 = \alpha_3 = 1/2\alpha_2$ into Eqs. (14)–(16), one obtains

$$\frac{d^2\alpha_1}{dt^2} = -\frac{2J}{3\alpha_1-1}\left(\frac{d\alpha_1}{dt} \right)^2, \tag{21}$$

$$\frac{d^2\alpha_2}{dt^2} = -\frac{J}{\frac{3}{2}\alpha_2-1}\left(\frac{d\alpha_2}{dt} \right)^2, \tag{22}$$

$$\frac{d^2\alpha_3}{dt^2} = -\frac{2J}{3\alpha_3-1}\left(\frac{d\alpha_3}{dt} \right)^2. \tag{23}$$

Their solutions are

$$\alpha_1 = \tfrac{1}{3}(1+2F) = \alpha_3, \tag{24}$$

$$\alpha_2 = \tfrac{2}{3}(1+2F), \tag{25}$$

$$F = [A_{20}(2J+3)t+1]^{(3/2J+3)}. \tag{26}$$

Then, we find the concentrations of A^1-, A^2-, and A^3-clusters are, respectively,

$$a_{1k}(t) = a_{3k}(t) = 18A_{20}F^{(2/3)J}(1+2F)^{-2}\left(1 - \frac{3}{1+2F} \right)^{k-1}, \tag{27}$$

$$a_{2k}(t) = 9A_{20}F^{-(2/3)J}(1+2F)^{-2}\left(1 - \frac{3}{2(1+2F)} \right)^{k-1}. \tag{28}$$

The asymptotic behaviors for the cluster-mass distributions can be easily found at a long time limit

$$a_{1k}(t) = a_{3k}(t) = \tfrac{9}{2} A_{20}[A_{20}(2J+3)t]^{-1-(3/2J+3)}\exp(-x),$$
(29)

$$a_{2k}(t) = \frac{9}{4} A_{20}[A_{20}(2J+3)t]^{-1-(3/2J+3)}\exp\left(-\frac{x}{2}\right).$$
(30)

They are valid in the scaling region

$$t \gg 1, \quad k \gg 1, \quad x = \frac{3k}{2}[A_{20}(2J+3)t]^{-(3/2J+3)} = \text{finite}.$$
(31)

These results indicate that for the given initial data the evolutions of cluster concentrations come in a scaling regime at a long time limit. The average mass of clusters increases with time. If we use $S(t)$ to denote the characteristic mass of an aggregation system, the cluster concentration $c_k(t)$ of aggregates at long time can be written in the following scaling form [5];

$$c_k(t) \simeq t^{-W} f(\mathbf{k/S(t)}), \quad S(t) \propto t^Z.$$
(32)

The $c_k(t)$ denotes the concentration of k mers of any type of species. Calculating the 0th and 1th moments of cluster-mass distribution, one can obtain the power-law expressions for the total number of clusters $N(t)$ and the total mass of clusters $M(t)$ at long time

$$N(t) = \sum_{k=1}^{\infty} c_k(t) \propto t^{-\lambda},$$
(33)

$$M(t) = \sum_{k=1}^{\infty} k c_k(t) \propto t^{-\mu}.$$
(34)

Substituting the scaling form (32) for the cluster-mass distribution into Eqs. (33) and (34), one finds the exponent relations

$$\lambda = W - Z, \quad \mu = W - 2Z.$$
(35)

From Eq. (29) or (30), one can find all these exponents

$$W = \frac{2J+6}{2J+3}, \quad Z = \frac{3}{2J+3}, \quad \lambda = 1, \quad \mu = \frac{2J}{2J+3}.$$
(36)

This result indicates that W, Z, and μ are dependent on the value of rate J. They are not universal constants independent of reaction processes.

In addition, from Eqs. (29) and (30) we see that the concentration of A^1- and A^3-clusters is much less than the one of A^2-clusters at long time. This difference is related to the initial mass distribution. From Eqs. (29), (30), and (34), one can easily find the initial total mass of A^1, A^2, and A^3 species. They are, respectively,

$$M_1(0) = 2A_{20}, \quad M_2(0) = 4A_{20}, \quad M_3(0) = 2A_{20}.$$

The initial total mass of A^2 species is two times as large as that of A^1 or A^3 species. The interplay between aggregation and annihilation makes the species with a larger initial total

mass decay slowly. It can be seen that initial mass distribution strongly affects the evolution behavior of reactants at long time.

(b) $\alpha_1 = \alpha_3 \neq \alpha_2$. This case is also exactly solvable. The initial conditions are still expressed by Eq. (17), but an additional condition $A_{10} = A_{30}$ is given. Substituting the condition into Eqs. (18) and (19), one obtains

$$2\alpha_1 - \alpha_2 = (2A_{10} - A_{20})t + 1 = 2\gamma_1 t + 1,$$
(37)

$$\left(\frac{d\alpha_1}{dt}\right)^2 \frac{d\alpha_2}{dt} = A_{10}^2 A_{20}(\alpha_1 + \alpha_2 - 1)^{-2J},$$
(38)

where $\gamma_1 = 1/2(2A_{10} - A_{20})$. If we assume $\gamma_1 > 0$, the asymptotic solutions for α_1 and $d\alpha_2/dt$ are given by

$$\alpha_1 \simeq \gamma_1 t, \quad \frac{d\alpha_2}{dt} \simeq C\gamma_1(1-2J)^{-1}(\gamma_1 t)^{-2J} \quad \text{at } t \gg 1,$$
(39)

where $C = A_{10}^2 A_{20} \gamma_1^{-3}$. Now, we discuss the three different cases, respectively.

(i) When $0 < J < 1/2$. One obtains scaling solutions

$$a_{1k} = a_{3k} = (\gamma_1 t^2)^{-1}\exp(-x),$$
(40)

$$a_{2k} = \gamma_1(1-2J)^2 C^{-1}(\gamma_1 t)^{2J-2}\exp(-y),$$
(41)

corresponding two different scaling variables

$$x = k(\gamma_1 t)^{-1} \quad \text{for } A^1\text{-}, A^3\text{-clusters},$$
(42)

$$y = (1-2J)C^{-1}k(\gamma_1 t)^{2J-1} \quad \text{for } A^2\text{-clusters}.$$
(43)

Then we have two kinds of exponents

$$W = 2, \quad Z = 1, \quad \lambda = 1, \quad \mu = 0 \quad \text{for } A^1\text{-}, A^3\text{-clusters},$$
(44)

$$W = 2 - 2J, \quad Z = 1 - 2J, \quad \lambda = 1, \quad \mu = 2J,$$
$$\text{for } A^2\text{-clusters},$$
(45)

and the total mass of A^1-, A^2- and A^3-clusters

$$M_1 = M_3 = \sum_{k=1}^{\infty} k a_{1k} = A_1(1-a_1)^{-2} = \frac{d\alpha_1}{dt} = \gamma_1 \quad \text{at } t \gg 1,$$
(46)

$$M_2 = \sum_{k=1}^{\infty} k a_{2k} = A_2(1-a_2)^{-2} = \frac{d\alpha_2}{dt}$$
$$= \gamma_1 C(\gamma_1 t)^{-2J} \quad \text{at } t \gg 1.$$
(47)

One can see that at a long time limit the mass of A^1- and A^3-clusters remains constant and the mass of A^2-clusters vanishes. This is natural. Because A^1 species and A^3 species both annihilate A^2 species at the same time, and make A^2 species decay quickly. After enough long time, A^1 species and A^3 species coalesces individually. But, from Eqs. (40) and (41), one can also see that when $t \gg 1$ and $k \ll (\gamma_1 t)^{1-2J}$ the concentration a_{2k} may be much greater than a_{1k}; small mass A^2-clusters dominate over the corresponding ones of A^1-clusters.

(ii) When $J=1/2$. The evolution behaviors of A^1- and A^3-clusters are still described by Eq. (40), while the evolutions of concentration, total number of clusters, and total mass for the A^2-clusters become, respectively,

$$a_{2k}(t)=C^{-1}t^{-1}(\ln t)^{-2}\exp(-y), \quad y=k(c\ \ln t)^{-1}, \tag{48}$$

$$N_2(t)=t^{-1}(\ln t)^{-1}, \tag{49}$$

$$M_2(t)=Ct^{-1}. \tag{50}$$

The logarithmic corrections appear in the A^2-clusters concentration and its 0th moment. This means that a small change of the annihilation rate may strongly change the decay of the A^2-species.

(iii) When $J>1/2$. It corresponds to an strong annihilation case. The α_2 becomes a decreasing function of time. If assuming α_2 approaches a steady stable value $\alpha_{2\infty}$ at the long time limit, one can obtain the asymptotic solution of α_2 by solving exactly Eqs. (37) and (38).

Substituting $(d\alpha_1/dt)=1/2(d\alpha_2/dt)+\gamma_1$ into Eq. (38) and setting $\Phi=3\alpha_2+2\gamma_1t+1$, one can obtain an implicit solution of α_2

$$\int_2^\Phi [F(V)-1]^{-1}dV=2\gamma_1t, \tag{51}$$

here

$$F=[F_1+(F_1^2-1)^{1/2}]^{1/3}]+[F_1-(F_1^2-1)^{1/2}]^{1/3}], \tag{52}$$

$$F_1=1+C3^32^{2J-2}\Phi^{-2J}, \tag{53}$$

and the asymptotic solution for $\alpha_2(t)$ at $t\gg1$

$$\alpha_2=\alpha_{2\infty}-\frac{3C}{2J-1}(\gamma_1t)^{1-2J}+\cdots, \tag{54}$$

$$\alpha_{2\infty}=1+\frac{1}{3}\int_2^\infty \{1-[F(V)-1]^{-1}\}dV. \tag{55}$$

Thus, we have a scaling solution for A^1- and A^3-clusters as Eq. (40) and the following nonscaling solution for A^2-clusters:

$$a_{2k}(t)\simeq\frac{3C\gamma_1(\gamma_1t)^{-2J}}{\alpha_{2\infty}(\alpha_{2\infty}-1)}\left(1-\frac{1}{\alpha_{2\infty}}\right)^k. \tag{56}$$

It turned out that a large annihilation rate results in a fast decay of A^2-clusters and the scaling description for the evolution behavior of A^2-clusters breaks down completely.

(c) $\alpha_1\neq\alpha_2\neq\alpha_3$. In the general asymmetrical case, Eqs. (18) and (19) are still satisfied exactly, but one cannot find an exact analytical solution for α_1, α_2, and α_3. When the initial masses of A^1 species and A^3 species, A_{10} and A_{30}, are much the same, one can imagine that species A^2 is divided into two parts. One part reacts with species A^1 and another with species A^3. If we again assume that $(A_{10}+A_{30}-A_{20})>0$, Eq. (18) can be recast into the following two equations:

$$\alpha_1-e_1\alpha_2=(A_{10}-e_1A_{20})t+1-e_1, \tag{57}$$

$$\alpha_2-e_2\alpha_2=(A_{30}-e_2A_{20})t+1-e_2, \tag{58}$$

where

$$e_1=\frac{A_{10}}{A_{10}+A_{30}}, \quad e_2=\frac{A_{30}}{A_{10}+A_{30}}. \tag{59}$$

From Eqs. (57), (58), and (19), one can obtain the asymptotic solutions for $\alpha_1(t)$, $\alpha_2(t)$, and $\alpha_3(t)$ at $t\gg1$

$$\alpha_1\simeq(A_{10}-e_1A_{20})t=\gamma_2t, \tag{60}$$

$$\alpha_3\simeq(A_{30}-e_2A_{20})t=\gamma_3t, \tag{61}$$

$$\alpha_2\simeq\overline{C}(1-2J)^{-1}(\gamma t)^{1-2J}, \tag{62}$$

where

$$\overline{C}=A_{10}A_{20}A_{30}\gamma^{-3} \quad \gamma^3=\gamma_2\gamma_3. \tag{63}$$

Corresponding cluster-mass distributions:
for A^1-clusters

$$a_{1k}(t)=(\gamma_2t^2)^{-1}\exp(-x_1) \quad x_1=k(\gamma_2t)^{-1}, \tag{64}$$

for A^3-clusters

$$a_{3k}(t)=(\gamma_3t^2)^{-1}\exp(-x_2) \quad x_3=k(\gamma_2t)^{-1}, \tag{65}$$

for A^2-clusters

$$a_{2k}(t)=\gamma(1-2J)^2C^{-1}(\gamma t)^{2J-2}\exp(-x_3)$$
$$x_3=(1-2J)\overline{C}^{-1}K(\gamma t)^{2J-1} \quad J<\tfrac{1}{2}, \tag{66}$$

$$a_{2k}(t)=(\overline{C}t)^{-1}(\ln t)^{-2}\exp(-x_4)$$
$$x_4=k(\overline{C}\ \ln t)^{-1} \quad J=\tfrac{1}{2}, \tag{67}$$

and when $J>1/2$, the nonscaling solution for A^2-clusters is similar to Eq. (56).

These approximation results coincide with those obtained by numerical computation to a certain degree of precision.

All preceding calculations for case (B) and case (C) are performed under the condition in which the value of every γ is greater than zero. When all γ are less than zero, the evolution behaviors of the A^2 species are scaling and the behaviors of A^1 and A^3 species are scaling or nonscaling, depending on the rate of the annihilation reaction. According to the definition of γ, the magnitude of γ corresponds to the difference of initial total mass among reactants. The effect of the value of γ on the evolution behavior of each species represents one of initial total mass. Therefore, we conclude that for a system with a three-species aggregation-annihilation reaction, the kinetic behaviors of the system depend conclusively on the aggregation rate, annihilation rate, and the initial mass distribution of each reactant.

III. MANY SPECIES AGGREGATION-ANNIHILATION MODEL WITH CONSTANT REACTION RATE

We suppose that there are n distinct species in a system. Each species aggregates itself, and any two kinds of distinct species annihilate each other. We assume again all aggrega-

tion rates are equal to one and all annihilation rates are equal to J. If n cluster-mass distributions are denoted by a_{lk}, ($l = 1,2,..,n$) the rate equations for the n species are given by

$$\frac{da_{lk}}{dt} = \sum_{i+j} a_{li}a_{lj} - 2a_{lk}\sum_{j=1}^{\infty} a_{lj}$$

$$+ J\sum_{m=1}^{n} \left(\sum_{p=q+k} a_{lp}a_{mq} - a_{lk}\sum_{j=1}^{\infty} a_{mj} \right),$$

$$m \neq l, \quad l = 1,2,\ldots,n, \quad m = 1,2,\ldots,n. \quad (68)$$

For the monodisperse initial conditions

$$a_{lk}(0) = A_{l0}\delta_{k1}, \quad A_{l0} = \text{const}, \quad l = 1,2,\ldots,n, \quad (69)$$

Eqs. (68) can be reduced in terms of ansatz

$$a_{lk}(t) = A_l(a_l)^{k-1}, \quad l = 1,2,\ldots,n. \quad (70)$$

Introducing new variables

$$\alpha_l = (1 - a_l)^{-1} \quad l = 1,2,\ldots,n, \quad (71)$$

into Eqs. (70), yields

$$\frac{d^2\alpha_l}{dt^2} = -\sum_{m=1}^{n} \frac{J}{\alpha_l + \alpha_m - 1} \frac{d\alpha_l}{dt} \frac{d\alpha_m}{dt},$$

$$m \neq l, \quad l = 1,2,\ldots,n, \quad (72)$$

with initial conditions

$$\alpha_l = 1, \quad \frac{d\alpha_l}{dt} = A_{l0} \quad \text{at } t = 0. \quad (73)$$

One can easily obtain one integral of Eqs. (72)

$$\prod_l \frac{d\alpha_l}{dt} = \prod_l A_{l0}\prod_l \prod_m (\alpha_l + \alpha_m - 1)^{-J/2}, \quad (74)$$

but cannot find each α_l as an explicit function of time in general.

Now, we consider the symmetrical case, i.e., $A_{l0} = A_0$, $\alpha_l = \alpha$ for all l. The solutions of $\alpha(t)$ and $a_k(t)$ for each species are, respectively,

$$\alpha(t) = \tfrac{1}{2}(1 + \Psi), \quad (75)$$

$$a_k(t) = 4A_0(1 + \Psi)^{-2}\Psi^{-(n-1/2)J}\left(\frac{\Psi - 1}{\Psi + 1}\right)^{k-1}. \quad (76)$$

$$\Psi = \{1 + [2 + (n-1)J]A_0 t\}^{[2/2+(n-1)J]}. \quad (77)$$

At long time limit the concentration is

$$a_k(t) \simeq 4A_0\{[2 + (n-1)J]A_0 t\}^{-1-[2/2+(n-1)J]}\exp(-x). \quad (78)$$

It is valid in the scaling region

$$k \gg 1, \quad t \gg 1,$$

$$x = 2k\{[2 + (n-1)J]A_0 t\}^{-[2/2+(n-1)J]} = (\text{finite}). \quad (79)$$

Setting $n = 2$, one obtains the solution for a symmetrical case in Ref. [23]. When $n = 3$, we have exponents

$$W = \frac{2+J}{1+J}, \quad Z = \frac{1}{1+J}, \quad \lambda = 1, \quad \mu = \frac{J}{1+J}. \quad (80)$$

Comparing the exponents in Eqs. (36) and (80), one can see that these exponents, except λ, are all dependent on the reaction rate, and for a given value of J, the value of Z in Eq. (36) is greater than Z in Eq. (80) and the value of μ in Eq. (36) is less than μ in Eq. (80). These differences among exponents represent the effect of different annihilation reaction models on the kinetic evolution behaviors of aggregates. In the general three-species aggregation-annihilation model, annihilation reaction occurs between any two distinct species. While in the special three-species aggregation-annihilation model in Sec. II, there is no annihilation reaction between the A^1 species and A^3 species. It turns out that the characteristic mass of clusters for each species in the general three-species aggregation-annihilation model increases slowly and the total mass of all clusters decayes faster than in the special three-species aggregation-annihilation model.

For general asymetric cases, it is difficult to find an analytical solution for each cluster concentration. The numerical computation for a system with three species or four species show that the kinetic behaviors of the system are very sensitive to the value of the reaction rate and the initial mass distribution.

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